C^0-topology of Hamiltonian homeomorphisms on Poisson manifold

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Abstract

In this paper, we define the topology of Hamiltonian homeomorphisms on regular Poisson manifold, and prove that Hameo(M) is a topological group and it is a normal subgroup of the Poisson Homeomorphisms, and show that the Hamiltonian homeomorphisms arising from the two norms coincide on the regular Poisson manifold.

Keywords: Poisson map, Hamiltonian homeomorphisms, Hamiltonian map, Hofer norm

Introduction

This paper is devoted to establishing the topology of Hamiltonian homeomorphisms of Hamiltonian dynamical system on Poisson manifold. Hamiltonian dynamical system plays an important role in differential geometry and mechanics, Hamiltonian homeomorphisms induced by this system are one of the main concerning objects. According to Klein's program: "Given a manifold and a transformation group acting on it, to investigate those properties of figures on that manifold which are invariant under transformations of that group". Studying symplectic geometry and Poisson geometry should research the group of symplectic diffeomorphisms and Poisson diffeomorphism. In symplectic dynamical system, this program was studied by many mathematicians. A manifold is symplectic if it is a smooth manifold and equipped with a closed non-degenerate differential 2-form, the Hamiltonian diffeomorphism is the time-1 map of the following dynamical system

$$i\left(\dot{f}_t\right)\omega = dF_t \quad . \tag{1}$$

Here $\dot{f}_t$ is defined by

$$\dot{f}_{t_0} = \frac{d}{dt} f_t\left(f_{t_0}^{-1}\right)_{|t=t_0} \quad . \tag{2}$$

When the manifold is symplectic, A bi-invariant metric was first discovered by Hofer on the group of compactly surpported symplectic diffeomorphism of (R^{2n},ω_0) (where ω_0 is the standard symplectic form) [1,2]. And Viterbo defined a bi-invariant metric on (R^{2n},ω_0) by generating function theory [3]. Polterovich extended it to more symplectic manifolds by the theory of pseudo holomorphic curves [8,9,17], and finally Lalonde and Mcduff to general symplectic manifold [4]. This metric plays an important role in studying symplectic topology, and has close relationship with symplectic capacity and symplectic rigity, many mathematicians have great work in this field, but there is few work on Poisson manifold. With the help of Casimir functions and the decomposition of Poisson manifold, we define a Hofer-type norm on Poisson manifold [10]. Oh and Muller introduced the notion of Hamiltonian limits of smooth Hamiltonian flows and constructed the C^0 concept of Hamiltonian diffeomorphisms, called Hamiltonian homeomorphisms, which forms a normal subgroup of the group of symplectic homeomorphisms[7]. Muller proved that the Hamiltonian homeomorphisms arising from the $L^{1,\infty}$ norm and the L^∞ norm coincide [6]. The contact topology is studied by Banyaga, Muller and Spaeth, they define the completion of the contact topology and the c^0 concept of contact diffeomorphisms [11,12,14-16]. With the help of C^0 contact topology, we can understand more about the contact manifold and contact dynamical systems. In order to detect more

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about the Poisson dynamical systems, we study the completion of Hamiltonian diffeomorphism on Poisson manifold and give the definitions of Hamiltonian homeomorphisms in the Poisson case. We use Hameo(M) to denote the sets of Hamiltonian homeomorphisms, the main results of this paper are the following:

Theorem 1 Hameo(M) is a topological group.

Theorem 2 The group of Hamiltonian homeomorphism $\mathrm{Hameo}(\mathrm{M})$ is path connected.

Theorem 3 Hameo(M) is a norm subgroup of the Poisson homeomorphisms Poisson(M).

Theorem 4 The Hamiltonian homeomorphism arising from the $L^{1,\infty}$ norm and the L^{∞} norm coincide on the Poisson manifold, that is

$$\mathrm{Hameo}_{\infty}(\mathrm{M}) = \mathrm{Hameo}_{1,\infty}(\mathrm{M}) \quad . \tag{3}$$

Organization of this paper: In the second part we will introduce some notions of Poisson manifold and give the construction of the Hamiltonian homeomorphisms. In the third part, we will give some lemmas and the proof of the main results.

Preliminaries

In this part, we will give some basic notions of Poisson manifold and Hofer metric, details can be found in [2,18,19]. Let (M,{ ˙ , ˙ }) be a Poisson manifold, i.e.,there exists a Poisson bracket{ ˙ , ˙ } on the smooth functions C^{∞}(M). For any function f, g, h$\in C^{\infty}$(M). it satisfies:

1.{f,g}= -{g,f},
2.{f,gh}=g{f,h}+h{f,g} (4)
3. {f,{g,h}}+{h,{f,g}}+{g,{h,f}}=0.

Definition 1 A smooth diffeomorphism φ: $\mathrm{M} \rightarrow \mathrm{M}$ is called a Poisson diffeomorphism if for all g, h $\in C^{\infty}$(M), we have

$$\varphi^{*}\{\mathrm{g},\mathrm{h}\}=\{\varphi^{*}g,\varphi^{*}h\} \quad . \tag{5}$$

Given h$\in C^{\infty}$(M), the Hamiltonian vector is defined by

$$X^{h} =\{ \cdot , h\}. \tag{6}$$

So next we consider the following Hamiltonian dynamical system

$$\dot{x}(t) = X_{H}(t,x) . \tag{7}$$

Let Cas(M)={f$\in C^{\infty}$(M): {f,g}=0, $\forall$ g$\in C^{\infty}$(M)}be the set of Casimir functions. We consider the time-dependent Hamiltonian functions C^{∞}([0,1]$\times$M,R). If the manifold is compact, or the function is compactly supported, then the flow of the Hamiltonian vector globally exists. In the following of this paper we always assume that M is closed. We denote by P(M),Ham(M) the set of such Hamiltonian flows and the set of time-1 map of such flows respectively.

Next we recall the definition of Hofer metric, For a smooth function $f \in C^{\infty}(M,R)$, we define its Hofer oscillation as following

$$\| f \|_{\infty} = \max_{x \in M} h(x) - \min_{x \in M} h(x) . \tag{8}$$

Now we define

$$\| f \| = \inf\{ \| f_1 \|_{\infty} \mid f = f_1 + f_2,\, f_2 \in Cas(M) \} . \tag{9}$$

If H_t is a Hamiltonian flow with some Hamiltonian function $h(x)$, we define its length to be

$$length(H_t) = \int_0^1 \| h_t(x) \| dt \tag{10}$$

the energy of $\phi \in Ham(M)$ is defined by

$$E(\phi) = \inf\{ length(H_t) \mid H_t \text{ is a hamiltonian flow ended with } \phi \}. \tag{11}$$

Then, we can define a function d as:

$$Ham(M) \times Ham(M) \rightarrow [0,\infty),$$
$$d(\varphi,\psi) = E(\varphi^{-1} \circ \psi), \tag{12}$$

for $\varphi,\psi \in Ham(M)$.

However, it is difficult to prove that it is indeed a really metric. In [10], it was shown that d is really a bi-invariant metric.

3 Construction of the Hamiltonian homeomorphisms

Let Homeo(M) be the group of homeomorphism of M with the c^0 topology, for $\phi,\varphi \in Homeo(M)$, their C^0 distance is defined by

$$\bar{d}(\phi,\varphi)$$
$$=\max\{\sup_x d(\phi(x),\varphi(x)),\sup_x d(\phi^{-1}(x),\varphi^{-1}(x))\},\quad (13)$$

where d is a c^0 distance on M induced by some Riemannian metric [7]. Then for given two continuous paths

$$\lambda,\mu:[0,1]\to \text{Homeo(M)},$$
$$\lambda(0)=\text{id},\ \mu(0)=\text{id}\ . \quad (14)$$

Their distance is defined by

$$\bar{d}(\lambda,\mu)=\max_{t\in[0,1]}\bar{d}(\lambda(t),\mu(t))\ . \quad (15)$$

Definition 2 Define Peo(M) to be

$$Poe(\text{M})=\overline{Poisson(\text{M})} \quad (16)$$

the C^0 closure of Poisson(M) in Homeo(M), and call Poe(M) the Poisson homeomorphism group.

We now define the Hamiltonian topology on the sets of Hamiltonian paths:

Definition 3 the Hamiltonian topology is generated by the colloections of subsets

$$\text{N}(\phi_H,\varepsilon_1,\varepsilon_2)$$
$$=\{\phi_{H'}\in P(\text{M})\big|\|\bar{H}\#\text{H}'\|<\varepsilon_1,\bar{d}(\phi_H,\phi_{H'})<\varepsilon_2\}. \quad (17)$$

For ε_1, ε_2 >0, and a smooth Hamiltonian path ϕ_H . The

Definition 4 The Hamiltonian topology on $p(M)$ is the topology induced by the following metric

$$d_{ham}(\phi_h^t,\phi_k^t)=\|(k-h)(t,\phi_h^t)\|+\bar{d}(\phi_h^t,\phi_k^t) \quad (18)$$

for $\phi_h^t,\phi_k^t\in p(M)$ .The Hamiltonian topology on Ham(M) to be the strongest topology such that the evaluation map

$$ev_1:P(\text{M})\to Ham(\text{M}) \quad (19)$$

is continuous.

Then from the definitions we have the following proposition.

Proposition 5 The Hamiltonian topology on P(M) is equivalent to the metric topology induced by d_{ham} .

Proposition 6 **The** left translation of the group P(M) are continuous, that is for each Hamiltonian path, the map

$$L_\lambda:P(\text{M})\to P(\text{M})$$
$$L_\lambda(\mu)=\lambda\mu\quad . \quad (20)$$

is continuous with respect to the Hamiltonian topology on P(M), the sets of the form:

$$\phi_H(\text{N}(id,\varepsilon_1,\varepsilon_2)),\varepsilon_1,\varepsilon_2>0 \quad (21)$$

makes a neighborhood basis at ϕ_H .

Proposition 6 gives rise the following corollary.

Corrolary 7 The evaluation map

$$ev_1:P(\text{M})\to Ham(\text{M}) \quad (22)$$

is an open map with respect to the Hamiltonian topology on Ham(M). For a Hamiltonian map $\phi\in Ham(\text{M})$ with the generating function H, the sets of the form:

$$ev_1(\text{N}(\phi_H,\varepsilon_1,\varepsilon_2)),\varepsilon_1,\varepsilon_2>0 \quad (23)$$

gives a neighborhood basis at ϕ in the Hamiltonian topology.

Following the methods of Oh and Muller [6,7], we now define the Hamiltonian homeomorphism. Consider the developing map

$$Dev:p(M)\to C^\infty([0,1]\times M,R),$$
$$Dev(\phi_h^t)=[h]\ , \quad (24)$$

Where $[h]$ denotes the equivalent class of the generating functions of ϕ_h^t, and the inclusion map:

$$\iota:p(M)\to Homeo^p(M,id),$$
$$\iota(\phi_h^t)=\phi_h^t\ , \quad (25)$$

where $Homeo^p(M,id)$ is the path in $Homeo(M)$ based at id .

We now consider the product maps defined as follows :

$$(Dev,\iota):p(M)\to C^\infty([0,1]\times M,R)\times Homeo^p(M,id)\ ,$$
$$\phi_h^t\to([h],\phi_h^t)\,. \quad (26)$$

Denote by Q the image of the above product map with product topology, and $\bar{Q}$ the closure of Q in $L^{1,\infty}([0,1]\times M,R)\times Homeo^p(M,id)$. From the definition of the Hamiltonian topology, the developing and inclusion maps can be extended to the the following continuous maps:

$$\overline{Dev}:\bar{Q}\to L^{1,\infty}([0,1]\times M,R)$$

$$\bar{\iota} : \overline{Q} \to Hoemeo^{p}(M, id), \tag{27}$$

and the evaluation map

$$ev_1 : Hemeo^{p}(M, id) \to Hemeo(M),$$
$$\lambda(t) \to \lambda(1) \tag{28}$$

can also be extended to

$$\overline{ev_1} : \mathrm{Im}\,age(ev_1) \to Homeo(M) \ . \tag{29}$$

Now we can give the definition of Hamiltonian homeomorphism.

Definition 8 Define the following sets

$$Hameo(M) = \{\psi \in Homeo(M) \big| \psi = \overline{ev_1}(\lambda), \lambda \in \mathrm{Im}\,age(\bar{\iota})\}, \tag{30}$$

and call any element in $Hameo(M)$ a Hamiltonian homeomorphismd.

4 Proof of the main results

Before we prove the main results we need the following lemmas which are very important to compute the generating functions for the Hamiltonian flows.

Definition 9 If h, k are smooth functions in $C^{\infty}([0,1]\times M, R)$ and $\vartheta \in Ham(M)$ we define the functions $\overline{h}, h\#k$ and h_{ϑ} as follows:

$$\overline{h}(t,x) = -h(t, \varphi_h^t(x))$$
$$h\#k(t,x) = h(t,x) + k(t, (\varphi_h^t)^{-1}(x)) \tag{31}$$
$$h_{\vartheta}(t,x) = h(t, \vartheta^{-1}(x)).$$

The following propositions and lemmas were proved in [2] in symplectic case and [10] for Poisson case.

Proposition 10 If h, k are smooth functions the following formulate hold true

$$\varphi_{\overline{h}}^t = (\varphi_h^t)^{-1}, \varphi_{h\#k}^t = \varphi_h^t \circ \varphi_k^t,$$
$$\vartheta \circ \varphi_h^t \circ \vartheta^{-1} = \varphi_{h\vartheta}^t, (\varphi_h^t)^{-1} o \varphi_k^t = \varphi_g^t, \tag{32}$$

where

$$g = \overline{h}\#k = (k-h)(t, \varphi_h^t). \tag{33}$$

Lemma 11 $\|\cdot\|$ defined above is a pseudo-norm and $Ham(M)$-invariant.

Lemma 12 let (M,{}) be a poisson manifold, and Π be a Poisson vector, on each symplectic leaf F,there is a well defined symplectic structure such that the inclusion map i: $F \to M$ is a poisson map, that is, there exists a 2-form $\omega|_F \in \Omega^2 F$ such that

$$\omega|_F(\mathrm{X}_f|_F, \mathrm{X}_g|_F) = \{\mathrm{f},\mathrm{g}\}|_F \quad . \tag{34}$$

For any function $f, g \in C^{\infty}(\mathrm{M})$, here $|_{.}$ denotes the restriction on F.

Lemma 13 let (M,{}) be a Poisson manifold, and Π be a Poisson vector, on each symplectic leaf F, For any functions $f, g \in C^{\infty}(\mathrm{M})$,we have

$$\{\mathrm{f},\mathrm{g}\}|_F = \{\mathrm{f}|_F, \mathrm{g}|_F\} \tag{35}$$

Lemma 14 If φ^t is the Hamiltonian flow of X_H, then For any functions $f, g \in C^{\infty}(\mathrm{M})$, there is:

$$(\varphi^t)^*\{\mathrm{f},\mathrm{g}\} = \{(\varphi^t)^* f, (\varphi^t)^* g\} \tag{36}$$

Having this, we can discuss the properties of Hamiltonian homeomorphisms.

Theorem 1. Hameo(M) is a topological group.

Proof :To prove that $Hameo(M)$ is a group, note that the Hofer norm in the Poisson case need involve the Casimir functions [10], so we need check that the inequalities of [6,7] still work in Poisson case. We first claim that the composition and inverse on the sets $Hameo(M)$ are defined as follows:

Suppose that $(\lambda,[h])\cdot(\mu,[f]) \in \overline{Q}$, then we define

$$(\lambda,[h])\cdot(\mu,[f]) = (\lambda \circ \mu, [h + f(\lambda^{-1})]),$$
$$(\lambda,[h])^{-1} = (\lambda^{-1}, [\overline{h}]). \tag{37}$$

Let $(\phi_{h_i}^t, [h_i])$ and $(\phi_{f_i}^t, [f_i])$ converge to $(\lambda,[h]), (\mu,[f])$ respectively in the Hamiltonian topology, and choose h, f represent $[h],[f]$.

First we have

$$\overline{d}(\lambda \circ \mu(t), \phi_{h_i}^t \circ \phi_{f_i}^t) \to 0,$$
$$\overline{d}(\lambda^{-1}, \phi_{h_i}^t)^{-1} \to 0, \tag{38}$$

as $i \to \infty$.

By the triangle inequality of the $\|\cdot\|$, we get

$$\int_0^1 \inf_{g\in Cas(M)} \left\| h_i \# f_i - (h + f(\lambda^{-1})) - g \right\|_\infty dt$$
$$\leq \int_0^1 \inf_{g\in Cas(M)} \left\| h_i - h - g \right\|_\infty dt +$$
$$\int_0^1 \inf_{g\in Cas(M)} \left\| [f_i(t,(\phi_{h_i}^t)^{-1}) - f(\lambda^{-1}) - g] \right\|_\infty dt \qquad (39)$$
$$\leq \int_0^1 \| h_i - h \| dt + \left\| f_i(t,(\phi_{h_i}^t)^{-1}) - f(t,(\phi_{h_i}^t)^{-1}) \right\|$$
$$+ \left\| f(t,(\phi_{h_i}^t)^{-1}) - f(t,\lambda^{-1}) \right\| dt.$$

By the assumption, the first two terms of equation (39) converge to zero as $i \to \infty$, we just need to compute the last term of equation (39), and

$$\int_0^1 \left\| f(t,(\phi_{h_i}^t)^{-1}(x)) - f(t,\lambda(x)) \right\| dt$$
$$\leq \int_0^1 \left\| f(t,(\phi_{h_i}^t)^{-1}(x)) - f_i(t,(\phi_{h_i}^t)^{-1}(x)) \right\|$$
$$+ \left\| f_i(t,(\phi_{h_i}^t)^{-1}(x)) - f_i(t,\lambda^{-1}(x)) \right\| \qquad (40)$$
$$+ \left\| f_i(t,\lambda^{-1}(x)) - f(t,\lambda^{-1}(x)) \right\| dt.$$

But by Lemma 3.21 of [7], and the fact that

$$\inf_{g\in Cas(M)} \| h_i - g \|_\infty \leq \| h_i - g \|_\infty . \qquad (41)$$

We know that the third term converges to zero and hence the composition is continuous.

Similarly, we still use Lemma 3.21 of [7] and the above fact,

$$\int_0^1 \left\| \bar{h}_i - h \circ \lambda \right\| dt \leq \int_0^1 \left\| h_i \circ (\phi_{h_i}^t) - h_i \circ \lambda \right\| + \left\| h_i \circ \lambda - h \circ \lambda \right\| dt$$
$$\to 0 \ . \qquad (42)$$

So the product map and the inverse map are continuous and we finish the proof.

Corollary 15 The group Hameo(M) contains all $C^{1,1}$ -Hamiltonian maps.

Proof: We can approximated the $C^{1,1}$ function by a sequence of smooth functions h_i such that

$$\int_0^1 \| h - h_i \| dt \to 0 \ \text{as} \ i \to \infty , \qquad (43)$$

On the other hand, the vector field $X_{H_i}(\mathrm{t,x})$ converges to $X_H(\mathrm{t,x})$ uniformly over $t \in [0,1]$, by the continuity theorem of ordinary differential equations for Lipschitz vector fields, the flow

$$\phi_{h_i}^t \to \phi_h^t \qquad (44)$$

in the C^0 -topology. And especially we have :

$$\phi_{h_i}^1 \to \phi_h^1 . \qquad (45)$$

So by the definition of Hamiltonian homeomorphism, we know that all $C^{1,1}$ -Hamiltonian maps are Hamiltonian homeomorphisms.

As we know, in the symplectic manifold, the group of Hamiltonian diffeomorphisms is path connected, since every Hamiltonian map can be connected with the identity map. Weinstein discovered the local connectedness of Symplectic diffeomorphisms by the symplectic neighbourhood theory [5,13]. Oh and Muller proved the Hamiltonian homeomorphism group is path connected. In the Poisson case, we proved that

Theorem 2 The group of Hamiltonian homeomorphism $\mathrm{Hameo}(\mathrm{M})$ is path connected.

Proof. To prove the connectedness, we should prove that every Hamiltonian Homeomorphism can be connected with the identity map.

By definition, suppose t $h \in Hameo(\mathrm{M})$, then there exists a sequence of Hamiltonian paths ϕ_{H_i} with Hamiltonian functions H_i, and satisfies that

$$\begin{aligned} &\phi_{H_i}^t \to \lambda, \\ &\mathrm{H}_i \to H, \\ &\lambda(0) = id, \lambda(1) = h. \end{aligned} \qquad (46)$$

Here λ is a continuous Hamiltonian path in the Hamiltonian homeomorphism group. Next we modify the paths by the following as in the methods of Oh and Muller in [7]:

$$\mathrm{t} \mapsto \phi_{H_i^s}^t = \phi_{H_i}^{st} \ . \qquad (47)$$

For all $s \in [0,1]$ and we have:

$$\begin{aligned} &\bar{d}(\phi_{H_i}^s, \phi_{H_{i'}}^s) \leq \bar{d}(\phi_{H_i}, \phi_{H_{i'}}) \to 0 , \\ &\| H_i^s - H_{i'}^s \| \leq \| H_i - H_{i'} \| \to 0, \\ &\text{as } i, i' \to 0 \ . \end{aligned} \qquad (48)$$

So the Hamiltonian functions H_i^s generating the path $\phi_{H_i^s}^t$ are a Cauchy sequence in the Hamiltonian topology. We denote by $(\lambda^s, \mathrm{H}^s)$ the new limit of $\phi_{H_i^s}^t$ and H_i^s, note that λ^s is just the path $\lambda(\mathrm{st})$, we have the following

$$\lambda(0) = id, \lambda(1) = h. \tag{49}$$

Since every Hamiltonian homeomorphism can be connected with the identity, we prove that $\mathrm{Hameo}(M)$ is path connected.

Proof of Theorem 2:

For $\vartheta \in \mathrm{Peo}(M), \varphi \in \mathrm{Hameo}(M)$, there exists diffeomorphism $\vartheta_i \in \mathrm{Poisson}(M), \phi_{h_i}^t \in P(M)$ such that

$$\phi_{h_i}^t \to \lambda \ ,$$

$$\int_0^1 \|h_i - h\| dt \to 0 \ , \tag{50}$$

as $i \to \infty$ and

$$\lambda(1) = \phi \tag{51}$$

is the norm which is defined before and which involves the Casimir functions. By Proposition 6 we need to show that:

$$\vartheta_i \circ \varphi_h^t \circ \vartheta_i^{-1} \to \vartheta \circ \lambda \circ \vartheta^{-1}. \tag{52}$$

$h_i(\vartheta_i)$ converges in the $L^{1,\infty}$ topology.

The first one is obvious, and the second term follows from the same argument as in the proof of Theorem 1.

$$\int_0^1 \|h_i(\vartheta_i) - h(\vartheta)\| dt \le \int_0^1 \|h_i(\vartheta_i) - h_i(\vartheta)\| + \|h_i(\vartheta) - h(\vartheta)\| dt \tag{53}$$
$$= \int_0^1 \|h_i(\vartheta_i) - h_i(\vartheta)\| + \|h_i - h\| dt.$$

We get that the above terms converge to zero, and finish the proof.

Remark 16 By the path connectedness, and the $L^{1,\infty}$ Approximation Lemma later, we can modify every topological Hamiltonian path to be boundary flat, we know that the concatenation of two topological Hamiltonian path is still a topological Hamiltonian path.

If we replace the $L^{1,\infty}$ -norm by the L^{∞} -norm in the definition of Hameo(M), we get another construction of Hamiltonian homeomorphisms. We denote by $\mathrm{Hameo}_{1,\infty}(M)$ and $\mathrm{Hameo}_{\infty}(M)$ respectively. In particular, we have

$$\mathrm{Hameo}_{\infty}(M) \subseteq \mathrm{Hameo}_{1,\infty}(M) \ . \tag{54}$$

By repeating the proof of Theorem 1, we can get the following results:

Theorem 17 $\mathrm{Hameo}_{\infty}(M)$ is also a topological group.

Theorem 18 $\mathrm{Hameo}_{\infty}(M)$ is a norm subgroup of $\mathrm{Hameo}_{1,\infty}(M)$.

Muller proved that $\mathrm{Hameo}_{1,\infty}(M)$ and $\mathrm{Hameo}_{\infty}(M)$ coincide in the symplectic case, we now prove this fact on the Poisson manifold.

Lemma 19 Let H be a smooth normalized Hamiltonian function generating the smooth Hamiltonian path

$$\phi_H : t \mapsto \phi_H^t. \tag{55}$$

Then given any $\varepsilon > 0$, there exists a smooth normalized Hamiltonian function $F : [0,1] \times M \to R$ such that F and hence ϕ_F is boundary, that is, there exists δ > 0 such that

$$F_t = 0 \text{ for } 0 \le t \le \delta,\ 1 - \delta \le t \le 1$$
$$\phi_F^0 = \phi_H^0 \ ,$$
$$\phi_F^1 = \phi_H^1, \tag{56}$$
$$\|F - H\|_{(1,\infty)} < \varepsilon$$
$$\bar{d}(\phi_F, \phi_H) < \varepsilon.$$

Proof: This is a consequence of Lemma 3.20 and 3.21 in [7] and the fact that

$$\inf_{g \in Cas(M)} \|h_i - g\|_{\infty} \le \|h_i - g\|_{\infty}. \tag{57}$$

The proof of the following lemmas are almost the same, we omit it here. The proof of these lemmas in the symplectic case can be found in Müller [6,7].

Lemma 20 Let H be a smooth normalized Hamiltonian function generating the smooth Hamiltonian path $\phi_H : t \in [0,1] \mapsto \phi_H^t$. Let $\varepsilon > 0$ be given. Then there exists a smooth normalized Hamiltonian function $F : [0,1] \times M \to R$ such that the following holds

$$\phi_F^0 = \phi_H^0 \ ,$$
$$\phi_F^1 = \phi_H^1, \tag{58}$$
$$\|F\|_{\infty} \le \|H\|_{1,\infty} + \varepsilon \ ,$$
$$\bar{d}\left(\phi_F, \phi_H^0\right) < \bar{d}\left(\phi_H, \phi_H^0\right) + \varepsilon.$$

where ϕ_H^0 denotes the constant path $t \mapsto \phi_H^0$.

Lemma 21 Let H be a smooth normalized Hamiltonian function generating the smooth Hamiltonian path $\phi_H : t \mapsto \phi_H^t$. There exists a positive constant C that depends only on H such that, given any $\varepsilon > 0$, there exists a smooth normalized Hamiltonian function

$F:[0,1]\times M\to R$ such that F and hence ϕ_F is boundary

$$\phi_F^0=\phi_H^0,$$
$$\phi_F^1=\phi_H^1,$$
$$\|F-H\|_\infty\le 2\|H\|_\infty+C_\varepsilon, \quad (59)$$
$$\|F\|_\infty\le 3\|H\|_\infty+C_\varepsilon,$$
$$\bar{d}(\phi_F,\phi_H)<\varepsilon.$$

Lemma 22 Given $0\leqslant a<b\leqslant 1$, and a smooth Hamiltonian H defined on [0,1]×M, we denote by $\zeta_{a,b}:[a,b]\to[0,1]$ the unique linear function with

$$\zeta(a)=0,\ \zeta(b)=1, \quad (60)$$

and by $H^{\zeta a,b}$ the reparameterized smooth Hamiltonian defined on $[0,1]\times M$. Then if H is normalized then so is $H^{\zeta a,b}$, and if H is boundary flat then again so is $H^{\zeta a,b}$, and we have the following

$$\|H^{\zeta a,b}\|_{1,\infty}=\|H\|_{1,\infty},$$
$$\|H^{\zeta a,b}\|_\infty=\frac{1}{b-a}\|H\|_\infty. \quad (61)$$

Proof of Theorem 3:

Since every Cauchy sequence H_i in the L^∞ topology is also a Cauchy sequence in the $L^{1,\infty}$ topology, we have $\mathrm{Hameo}_{1,\infty}(M)\supseteq\mathrm{Hameo}_\infty(M)$.

Now we prove the converse. Let $\phi\in\mathrm{Hameo}_{1,\infty}(M)$, by definition, there exists a sequence (ϕ_{H_i},H_i) of smooth Hamiltonian functions H_i generating the smooth Hamiltonian paths

$$\int_0^1\left\|\bar{h}_i-h\circ\lambda\right\|dt\le\int_0^1\left\|h_i\circ(\phi_{h_i}^t)-h_i\circ\lambda\right\|dt$$
$$+\int_0^1\|h_i\circ\lambda-h\circ\lambda\|dt\to 0 \quad (62)$$

with

$$\phi_{H_i}^0=id$$

such that

$$\left\|\bar{H}_j\# H_j\right\|=\|H_i-H_j\|_{1,\infty}\to 0 \text{ as } i,j\to\infty, \quad (63)$$

$\phi_{H_i}^1$ is a Cauchy sequence, its limit is ϕ.

Now using above lemmas and the procedure in [6], we can modify this sequence to be a L^∞ Cauchy sequence , Define

$$H_0=0,\phi_i=\phi_{H_i}{}^1,$$
$$K_i=\overline{H_{i-1}}\# H_i. \quad (64)$$

Applying lemma to each K_i , we get a sequence of L_i, use lemma ,we get a sequence of boundary flat functions M_i satisfying their requirements there, and using Lemma to reparameterize M_i on small segments $[t_{i-1},t_i]$, here t_i is defined as

$$t_i=1-\frac{1}{2^i}. \quad (65)$$

Now we can concatenate the new function on the segments $[t_{i-1},t_i]$ since they are boundary flat and we can prove that

$$\|F_i-F_{i-1}\|_\infty\le\frac{1}{2^i},$$
$$\bar{d}(\phi_{F_{i-1}},\phi_{F_i})\le\frac{1}{2^i}. \quad (66)$$

This finishes the proof.

Remark 23 When the Poisson manifold is symplectic, that is, there is only one leaf, in this case the Hofer norm is just the one defined by Hofer, and the Hamiltonian homeomorphism is just the same with the construction of Oh and Muller's.

Remark 24 Theorem 1 holds not only for regular manifold, but also for many other manifold, for example, when the rank of the Poisson manifold is not zero, or the symplectic leaves are always open or always closed.

5 Conclusion

This paper first gives the conception of Hamiltonian homeomorphisms on Poisson manifold, and proves that it is a topological group, we also establish some approximation lemmas in Poisson dynamical systems, and finally extend Muller's results from symplectic case to Poisson case.

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